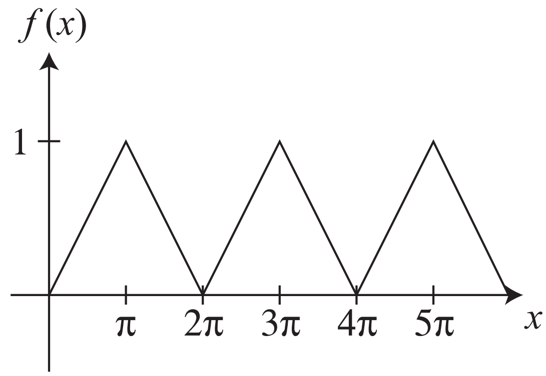
Mathematical Methods for Optical Physics and Engineering
Additional solved exercises, Chapter 6

1. A triangle wave function is a periodic function $f(x)$ such that $f(x + 2\pi) = f(x)$; a few cycles of such a function are illustrated below:



Determine, using distributions as necessary, the first and second derivatives of this function for all x. (Hint: the fact that the function repeats indicates that you really only need to describe these derivatives for one representative period.)

Here we have to find a representation of the function $f(x)$ first, before we can take derivatives. For $0 \leq x < 2\pi$, we may write

$$f(x) = \begin{cases} \frac{x}{\pi}, & 0 \leq x < \pi, \\ \frac{2\pi - x}{\pi}, & \pi \leq x < 2\pi. \end{cases}$$

The function is continuous but the derivative will be piecewise continuous. Direct differentiation gives

$$f'(x) = \begin{cases} \frac{1}{\pi}, & 0 \leq x < \pi, \\ -\frac{1}{\pi}, & \pi \leq x < 2\pi. \end{cases}$$

The derivative of the sawtooth is simply a square wave! If we restrict ourselves to $0 \leq x < 2\pi$ for clarity, the derivative of a unit upward step is a delta function, which means we may write

$$f''(x) = \frac{2}{\pi}\delta(x) - \frac{2}{\pi}\delta(x - \pi).$$

This pattern of deltas will repeat in every 2π interval. We may explicitly write the second derivative as an infinite series,

$$f''(x) = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\delta(x - 2n\pi) - \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\delta(x - (2m+1)\pi).$$

2. Calculate the values of the following integrals:

   (a) $I_1 = \int_{-\infty}^{\infty}\delta(x - \pi/6)\sin(3x + \pi)dx$,
   (b) $I_2 = \int_{2}^{\infty}\delta''(x - 4)\log(2x)dx$,
   (c) $I_3 = \int_{0}^{5}\delta(x + 5)[x^2 + 2x + 2]dx$,
   (d) $I_4 = \int_{-2}^{3}\delta'(x + 1)[x^4 + 2x^2 + x]dx$.

   The results can be found using basic principles of delta functions; we have

   (a) $I_1 = -1$.
   (b) $I_2 = -1/16$.
   (c) $I_3 = 0$. The delta is centered outside of the integration range!
   (d) $I_4 = 7$.

3. Using the sifting property of the delta function, and delta sequences if necessary, determine the behavior of the distribution

$$\sin(2x)\delta''(x) + 2\cos(2x)\delta'(x).$$

   This is an example where it is important to include the test function $f(x)$ in our calculation – if we simply integrate the function given, we get zero. However, if we integrate the given distribution (call it $D(x)$) with a test function, we find that

$$\int_{-\infty}^{\infty} D(x)f(x)dx = 2f'(0).$$

4. Simplify the expression for the distribution $\delta[(x+1)(x^2+x-6)]$.

   Here we must apply the composition property, where the function $g(x) = (x+1)(x+3)(x-2)$. The result is
$$\delta[g(x)] = \frac{\delta(x+1)}{6} + \frac{\delta(x+3)}{10} + \frac{\delta(x-2)}{15}.$$

5. We can derive a step function from a delta sequence, as well as the other way around. Demonstrate that
$$\lim_{n\to\infty} \int_{-\infty}^{x} \delta_n(x')dx' = S(x),$$
   where
$$\delta_n(x) = \frac{n}{2}\exp[-n|x|].$$

   This is relatively straightforward to demonstrate, provided one remembers that, since $\delta_n(x)$ is a piecewise function, the integral is piecewise as well! One finds that
$$\int_{-\infty}^{x} \delta_n(x')dx' = \begin{cases} \frac{1}{2}e^{nx}, & x < 0, \\ 1 - \frac{1}{2}e^{-nx}, & x \geq 0. \end{cases}$$

   In the limit $n \to \infty$, the function becomes zero for $x < 0$ and becomes unity for $x \geq 0$, with a value $1/2$ at $x = 0$. This is the behavior of the step function we expect!

6. Simplify the expression for the distribution,
$$H(x) = \sin(x)\delta(\sin^2 x - 1/2),$$
   by writing it as an infinite sum of "simple" delta functions.