Mathematical Methods for Optical Physics and Engineering
Additional solved exercises, Chapters 1 and 2

1. Consider the vectors $\mathbf{A} = 6\hat{\mathbf{x}} + 3\hat{\mathbf{z}}$ and $\mathbf{B} = 5\hat{\mathbf{x}} + 2\hat{\mathbf{z}}$. Calculate the scalar product of these vectors using Eq. (1.38). Also, determine the magnitudes of the vectors and the angle between them, and confirm that the scalar product satisfies Eq. (1.37).

   The scalar product is readily found to be $\mathbf{A} \cdot \mathbf{B} = 36$. The magnitudes of the two vectors are $|\mathbf{A}| = \sqrt{45}$ and $|\mathbf{B}| = \sqrt{29}$. The vector $\mathbf{A}$ makes an angle of 26.5° with the x-axis, while the vector $\mathbf{B}$ makes an angle 21.8°. The angle between the two vectors is found to be 4.7°; we can compare this with $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$. Solving for θ, we readily find agreement.

2. Consider the vectors $\mathbf{A} = 3\hat{\mathbf{x}} + 2\hat{\mathbf{y}}$, $\mathbf{B} = \hat{\mathbf{y}} + \hat{\mathbf{z}}$, and $\mathbf{C} = \hat{\mathbf{x}}$. Calculate the triple scalar product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and the triple vector product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ of these vectors.

   These products can be calculated straightforwardly; the results are

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= 2, \\ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= -2\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 3\hat{\mathbf{z}}. \end{aligned}$$

3. Suppose that we may express a vector $\mathbf{U}$ in terms of three unit vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ *that are not orthogonal*, i.e. $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} \neq 0$, and so forth, so that

$$\mathbf{U} = u\hat{\mathbf{u}} + v\hat{\mathbf{v}} + w\hat{\mathbf{w}}.$$

   Using properties of vector multiplication, determine expressions for u, v, and w in terms of $\mathbf{U}$, $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$.

   The key here is the observation that the cross-product of two vectors, say $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, is perpendicular to both of them, i.e. $(\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{u}} = 0$. We therefore take the dot product of the above equation with $\hat{\mathbf{v}} \times \hat{\mathbf{w}}$, which leaves us

$$(\hat{\mathbf{v}} \times \hat{\mathbf{w}}) \cdot \mathbf{U} = u(\hat{\mathbf{v}} \times \hat{\mathbf{w}}) \cdot \hat{\mathbf{u}}.$$

Solving for u, we get

$$u = \frac{(\hat{\mathbf{v}} \times \hat{\mathbf{w}}) \cdot \mathbf{U}}{(\hat{\mathbf{v}} \times \hat{\mathbf{w}}) \cdot \hat{\mathbf{u}}}.$$

Similarly, we may readily find that

$$v = \frac{(\hat{\mathbf{u}} \times \hat{\mathbf{w}}) \cdot \mathbf{U}}{(\hat{\mathbf{u}} \times \hat{\mathbf{w}}) \cdot \hat{\mathbf{v}}},$$

$$w = \frac{(\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \mathbf{U}}{(\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}}}.$$

4. We consider the path integral of the vector function $\mathbf{v}(x, y) = y\hat{\mathbf{x}} + xy\hat{\mathbf{y}}$, where the path C is the unit circle in the xy-plane centered on the origin. Determine the value of

   (a) $\oint_C \mathbf{v} dr$,

   (b) $\oint_C \mathbf{v} \cdot d\mathbf{r}$.

   (Parameterize the curve by choosing $x = \cos\theta$, $y = \sin\theta$.)

   Please be sure to check the book errata, as the expression for dr is incorrect in the text! With the proper expression, we have $dr = d\theta$. On substitution of the appropriate parameters, we find that

$$\oint_C \mathbf{v} dr = 0.$$

   For the second integral, $d\mathbf{r} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}$. The integral in this case takes on the form

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = -\pi.$$

5. It is somewhat surprising to note that almost any parameterization of a path integral will work, provided it accurately characterized the path. Repeat Example 2.1 with the parameterization $\mathbf{r}(t) = (2\log(t)\hat{\mathbf{x}} + 3\log(t)\hat{\mathbf{y}})/\log(2)$ along C_1, with $t : 1 \to 2$, and $\mathbf{r}(s) = 2s^3\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$ along C_2, with $s : 1 \to 0$, and show that the result is the same.

   With the first part of the parameterization, we find that

$$\frac{d\mathbf{r}(t)}{dt} = \frac{2}{t}\hat{\mathbf{x}} + \frac{3}{t}\hat{\mathbf{y}},$$

   with $t : 1 \to 2$. Plugging in and performing the integration, one gets $35/2$ for this leg.

   For the second leg,

$$\frac{d\mathbf{r}(s)}{ds} = 6s^2\hat{\mathbf{x}},$$

   with $s : 1 \to 0$. On substitution, one find that this integral evaluates to -6, which brings the total integral to $23/2$, in agreement with the book result (which as a minus sign error).

6. Demonstrate that the divergence theorem is satisfied in a cube of unit side with edges along the coordinate axes for the function

$$\mathbf{v} = x\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + x^2\hat{\mathbf{z}}.$$

Both the surface and the volume integral should result in a value of 2.

7. Demonstrate that Stokes' theorem is satisfied for a square curve of unit side with edges along the x and y coordinate axes, for the function

$$\mathbf{v} = xy^3\hat{\mathbf{x}} + y^2x^2\hat{\mathbf{y}}.$$

Both the surface and the path integral should result in a value of $-1/6$, provided the path integral is taken along the $+x$-axis.